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D-branes as a bubbling Calabi-Yau

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Abstract: We prove that the open topological string partition function on a D-brane configuration in a Calabi-Yau manifold X takes the form of a closed topological string partition function on a different Calabi-Yau manifold X_b. This identification shows that the physics of D-branes in an arbitrary background X of topological string theory can be described either by open+closed string theory in X or by closed string theory in X_b. The physical interpretation of the "bubbling" Calabi-Yau X_b is as the space obtained by letting the D-branes in X undergo a geometric transition. This implies, in particular, that the partition function of *closed* topological string theory on certain bubbling Calabi-Yau manifolds are invariants of knots in the three-sphere.

Keywords: Gauge-gravity correspondence, Topological Strings, D-branes.

Contents

1. Introduction and conclusion

D-branes in a given vacuum of string theory have two alternative descriptions; either in terms of open strings or in terms of closed strings. This basic observation motivates the existence of a duality between open+closed string theory in the given vacuum and closed string theory in the vacuum where the D-branes have been replaced by a non-trivial geometry with fluxes.[1]

In this paper we give a very concrete realization of open/closed duality. We find an explicit relation between the partition function of open+closed topological string theory in a given Calabi-Yau X and the partition function of closed topological string theory in another "bubbling" Calabi-Yau X_b:

$$Z_{o+c}(X) = Z_c(X_b). \tag{1.1}$$

[1]This type of duality is often studied in a low energy, decoupling limit where the open+closed string theory on one side of the duality reduces to a gauge theory. Taking the same limit on the purely closed string side of the duality in effect replaces the asymptotic geometry of the original vacuum by a new asymptotic geometry. The AdS/CFT correspondence is the prototypical example of such a "low energy" open/closed duality.

The physical interpretation of X_b is as the background obtained by replacing the D-branes in X by "fluxes" when the D-branes undergo a geometric transition. This equality shows that the physics of D-branes in an arbitrary background X of topological string theory can be described either by open+closed string theory in X or by closed string theory in X_b.

The identification of the open+closed partition function in X with the closed string partition function in X_b does not rely on knowing explicitly the exact answer for the partition functions, which is why the result applies in great generality. The result relies on being able to write the open string partition function in terms of the open Gopakumar-Vafa (GV) invariants [1, 2] and the closed string partition function in terms of the closed Gopakumar-Vafa (GV) invariants [3]. As reviewed in section 2, such a parametrization of the open string partition function is possible whenever the world-volume geometry of the D-branes defining the open string theory has a non-trivial first Betti number $b_1(L)$, where L is the cycle that the D-branes wrap. It is for such open string theories that we can explicitly show that they are completely equivalent to a closed string theory on a "bubbling" Calabi-Yau space X_b.

In order to completely determine the open string partition function in a Calabi-Yau X we must supply the open GV invariants in X *and* the holonomy of the gauge field on the branes. Since the holonomy of the gauge field encodes[2] the "position" of the branes, the open string amplitude depends on the holonomy. Following [4], we encode the data about the holonomy matrix in a Young tableau,[3] labeled by R. Given this data we prove that the open+closed string partition function on X can be rewritten precisely as a closed string partition function on another Calabi-Yau X_b. Namely, the open string partition function in X can be written as a closed string instanton expansion on X_b, which is what the closed string partition function in topological string theory computes.

We find an explicit formula relating the closed GV invariants in X_b to the open+closed GV invariants in X and the holonomy of the gauge field living on the D-branes. As we recall in section 2 the GV invariants are a collection of *integers* in terms of which the topological string theory partition function on a Calabi-Yau manifold can be written down to all orders in perturbation theory. The formula we find takes the integer open and closed GV invariants in X together with the holonomy of the gauge field labeled by the Young tableau R and relates them to a new set of integers, which are precisely the closed GV invariants in another space X_b!

By using the relation we obtain between the closed GV invariants in X_b and the open+closed GV invariants in X combined with the holonomy of the gauge field, we can explicitly identify the closed string partition function in X_b with the open+closed string partition function in X. This computation demonstrates that the physics of D-branes in X is completely equivalent to *closed* string physics in X_b. This gives a way to explicitly construct open/closed dualities even when the explicit expressions for the partition functions are not known. It allows us to relate open string theory in X with closed string theory in X_b.

[2]The position is defined only up to Hamiltonian deformations, which are gauge symmetries of the A-model open string field theory.

[3]See section 3 for details.

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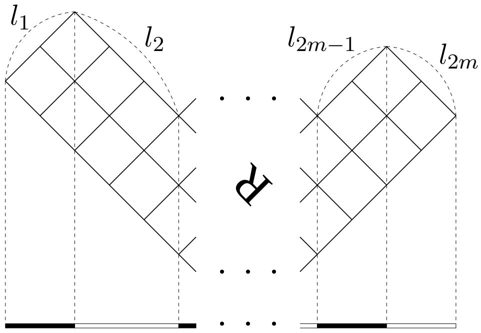

Figure 1: The Young tableau R, shown rotated, is specified by the lengths l_I of all the edges. Equivalently, l_I denote the length of the black and white regions in the Maya diagram.

The topology of X_b depends on the topology of X and on the shape of the Young tableau R. If we parametrize the Young tableau by using the following coordinates[4]
then we find that $b_2(X_b) = b_2(X) + 2m$, where b_2 is the second Betti number of the manifold. The size of the extra $2m$ two-cycles created by replacing the branes by "flux" is given by $t_I = g_s l_I$, with $I = 1, \ldots, 2m$, where l_I are the coordinates of the Young tableau in figure 1. The appearance of the extra cycles has a simple physical intepretation. The branes in X can undergo a geometric transition and be replaced by fluxes. Fluxes in topological string theory correspond precisely to non-trivial periods of the complexified Kähler form. In this picture, the original branes disappear and leave behind a collection of non-contractible cycles on which their flux is supported. Therefore, the Calabi-Yau X_b captures the backreacted geometry produced by the D-branes in X. It is this picture that warrants the description of X_b as a bubbling Calabi-Yau.

An interesting application of these results is to knot invariants in S^3. On the one hand, knot invariants in S^3 are captured by the expectation value of Wilson loops in Chern-Simons theory in S^3 [5]. On the other hand, as shown in [4], a Wilson loop operator in $\mathrm{U}(N)$ Chern-Simons theory on S^3 — which is labeled by a representation R and a knot α — is described by a configuration of D-branes or anti-branes in the resolved conifold geometry (see [4] for the details of the brane and anti-brane configuration).

Since we can now relate the open+closed GV invariants of a brane configuration in the resolved conifold to the closed GV invariants in X_b, we arrive at the representation of knot invariants in terms of closed GV invariants in X_b. This relation was already established in [4] for the case of the unknot and for arbitrary representation R, where it was shown that these knot invariants are captured by the *closed* topological string partition function on certain bubbling Calabi-Yau manifolds. Therefore, as a corollary of the results in this paper and those in [4] we find a novel representation of knot invariants for arbitrary knots in S^3 in terms of closed GV invariants of bubbling Calabi-Yau manifolds X_b!

An interesting recent development in the application of topological strings to knot theory is the so-called categorification program [6, 7]. The idea is to use the BPS Hilbert

[4]Informally, l_{odd} is the number of rows in the tableau with the same number of boxes while l_{even} is the number of columns in the tableau with the same number of boxes.

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space associated with open strings on the branes realizing knots to define more refined invariants than knot polynomials. Our proposal in [4] and in this paper is that these branes can undergo a geometric transition to bubbling Calabi-Yau manifolds. We are then tempted to contemplate that the BPS Hilbert space associated with *closed* strings on the bubbling Calabi-Yau manifolds could be used define new knot invariants.

The results in this paper confirm the expectation that whenever we have many branes in a given open+closed string theory, we have a dual description in terms of pure closed string theory in the backreacted geometry, where branes are replaced by non-trivial geometry with fluxes. It would be very interesting to extend the ideas in this paper to physical string theory. Learning how to rewrite open string theory in a given background as a closed string theory in a different background would be tantamount to deriving open/closed dualities in the physical theory.

This paper focuses on geometric transitions, namely on transitions of D-branes into pure geometry with flux. Another interesting phenomenon found in the study of Wilson loops in $\mathcal{N}=4$ Yang-Mills and Chern-Simons theory is that fundamental strings describing Wilson loops can puff up into D-branes. Just like for geometric transitions one may expect that the transition between strings and D-branes occurs more generally. The forthcoming paper [8] will discuss a large class of such transitions in the topological string setting.

The plan for the rest of the paper is as follows. In section 2 we give a brief summary of the physical origin of open and closed GV invariants and how they characterize the topological string partition function for open and closed strings. In section 3 we show that the partition function of open+closed string theory in a Calabi-Yau X is equal to the closed string partition function in a bubbling Calabi-Yau X_b. We argue that X_b is the space obtained by letting the D-branes in X undergo a geometric transition. In section 4 we study the geometric transitions proposed in this paper in the context of toric Calabi-Yau manifolds and show that the transitions we propose can be explicitly exhibited. The appendices contain the derivation of various formulas appearing in the main text.

2. GV invariants in a nutshell

The topological string partition function in X computes certain F-terms [9, 10, 1] in the effective action obtained by compactifying ten dimensional string theory on X. The physical origin of GV invariants stems from the observation in [1−3] that these higher derivative terms in Type IIA string theory do not depend on the string coupling constant, and can also be computed using an index that counts the BPS spectrum of wrapped membranes in an M-theory compactification on X.

The upshot is that the topological string amplitudes exhibit hitherto unknown integrality properties. Remarkably, the partition function can be computed to all orders in perturbation theory in terms of the integral invariants [1−3] associated to a given Calabi-Yau.

Closed GV invariants. The closed string partition function Z_c on X computes the supersymmetric completion of the following higher derivative term in the four dimensional effective action[5]

$$F(g_s,t)R_+^2, \tag{2.1}$$

where:

$$F(g_s,t)=\sum_{g=0}^{\infty}F_g(t)g_s^{2g-2} \qquad \text{and} \qquad Z_c(g_s,t)=\exp(F(g_s,t)). \tag{2.2}$$

$F_g(g_s,t)$ is the genus g topological string free energy and g_s is the topological string coupling constant. The complex scalar fields $\vec{t}\equiv(t_1,\ldots,t_{b_2(X)})$ in the physical theory parametrize the "size" of the various two cycles in X

$$t_a=\int_{\Sigma_a}\mathcal{J}, \tag{2.3}$$

where Σ_a are an integral basis of $H_2(X,\mathbf{Z})$ and $\mathcal{J}$ is the complexified Kähler form.

It has been argued by Gopakumar and Vafa [11, 12] that $F(g_s,t)$ can be computed in terms of integer invariants $n_g^{\vec{Q}}\in\mathbf{Z}$, where $g\in\mathbf{Z}_{\geq 0}$ and $\vec{Q}\equiv(Q_1,Q_2,\ldots,Q_{b_2(X)})\in\mathbf{Z}^{b_2(X)}$. These integers $n_g^{\vec{Q}}$ are called invariant because they do not change under smooth complex structure deformations of X; they define an index. Roughly speaking, $n_g^{\vec{Q}}$ counts[6] the number of BPS multiplets arising from membranes wrapping the class $\vec{\Sigma}\cdot\vec{Q}\in H_2(X,\mathbf{Z})$. As shown in [11, 12] a one-loop diagram with membranes running in the loop precisely generates the term (2.1) in the four dimensional effective action. By comparing the one-loop diagram with (2.1) one finds that [11, 12]:

$$Z_c(g_s,t)=M(q)^{\frac{\chi(X)}{2}}\cdot\exp\left(\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}\,[n]^{2g-2}\sum_{\vec{Q}}n_g^{\vec{Q}}e^{-n\vec{Q}\cdot\vec{t}}\right). \tag{2.4}$$

$[n]\equiv q^{n/2}-q^{-n/2}$ is a q-number, where $q\equiv e^{-g_s}$ and $\chi(X)$ is the Euler characteristic[7] of X. The function

$$M(q)=\prod_{m=1}^{\infty}\frac{1}{(1-q^m)^m} \tag{2.5}$$

is the MacMahon function, and arises from the contribution of $D0$-branes — or eleven dimensional momentum — running in the loop. From the world-sheet point of view, this is the contribution from constant maps from the world-sheet to X [11, 13].

Knowledge of the closed GV invariants $n_g^{\vec{Q}}$ in X determines using (2.4) the closed topological string partition function in X to all orders in perturbation theory.

[5] In writing this term we have already turned on a graviphoton field strength background $F=g_s$, where g_s is the topological string coupling constant. R_+ is the self-dual part of the curvature.

[6] g encodes the quantum number under $\mathrm{SU}(2)_L$, a subgroup of the rotation group in the four non-compact directions.

[7] For a compact Calabi-Yau manifold, $\chi(X)/2$ is the number of Kähler moduli minus the number of complex structure moduli.

Open GV invariants. The open string partition function Z_o in X computes the supersymmetric completion of the following term in the two dimensional effective action that arises by wrapping P $D4$-branes on a special Lagrangian submanifold[8] $L \subset X$

$$F(g_s, t, V)R_+, \tag{2.6}$$

where:

$$F(g_s, t, V) = \sum_{g=0}^{\infty}\sum_{h=1}^{\infty} F_{g,h}(t, V) g_s^{2g-2+h} \quad \text{and} \quad Z_o(g_s, t, V) = \exp(F(g_s, t, V)). \tag{2.7}$$

$F_{g,h}(g_s, t, V)$ is the topological string free energy on a genus g Riemann surface with h boundaries, with the boundary conditions specified by a Lagrangian submanifold L, which gives rise to BRST-invariant boundary conditions. V is the $\mathrm{U}(P)$ holonomy matrix that arises by integrating the gauge field on the $D4$-branes along the generator of $H_1(L, \mathbf{Z})$. It corresponds to a complex scalar[9] field in the effective two dimensional theory living on the $D4$-branes.

It was shown in [1, 2] that these terms also arise at one-loop by integrating out BPS states that end on the $D4$-branes. By comparing the one-loop computation with (2.6) one arrives at the following expression [1, 2]:

$$Z_o(g_s, t, V) = \exp\left(\sum_{n=1}^{\infty}\sum_{\vec{k}} \frac{1}{n}\frac{1}{z_{\vec{k}}} f_{\vec{k}}(q^n, e^{-n\vec{Q}\cdot\vec{t}})\mathrm{Tr}_{\vec{k}} V^n\right). \tag{2.8}$$

In the computation the symmetric group S_k plays a prominent role. $\vec{k} = (k_1, k_2, \ldots)$ labels a conjugacy class $C(\vec{k})$ of S_k since $\vec{k}$ corresponds to a partition of k:

$$k = \sum_j j k_j. \tag{2.9}$$

The integers $z_{\vec{k}} \equiv \prod_j k_j! j^{k_j}$ encode the number of permutations $N(C(\vec{k}))$ in the conjugacy class $C(\vec{k})$, which is given by $N(C(\vec{k})) = k!/z_{\vec{k}}$. Also:

$$\mathrm{Tr}_{\vec{k}} V \equiv \prod_j (\mathrm{Tr} V^j)^{k_j}. \tag{2.10}$$

The function $f_{\vec{k}}(q, e^{-\vec{Q}\cdot\vec{t}})$ in (2.8) can be written in terms of the open GV invariants $\widehat{N}_{Rg\vec{Q}} \in \mathbf{Z}$ [1, 2]:

$$f_{\vec{k}}(q^n, e^{-n\vec{Q}\cdot\vec{t}}) = \sum_{g=0}^{\infty} [n]^{2g-2} \prod_{j=1}^{\infty} [nj]^{k_j} \sum_{\vec{Q}}\sum_{R} \chi_R(C(\vec{k}))\widehat{N}_{Rg\vec{Q}} e^{-n\vec{Q}\cdot\vec{t}}. \tag{2.11}$$

As before $[a] \equiv q^{a/2} - q^{-a/2}$, R is a representation of S_k and[10] of $\mathrm{U}(P)$ labeled by a Young tableau R and $\chi_R(C(\vec{k}))$ is the character in the representation R of S_k for the conjugacy class $C(\vec{k})$. Roughly speaking, the integers $\widehat{N}_{Rg\vec{Q}}$ count[11] the number of BPS multiplets

[8] In order not to clutter the formulas and obscure the physics, we will assume that $b_1(L) = 1$ in writing the formulas. It is straightforward to write the corresponding formulas for $b_1(L) \geq 1$.

[9] We recall that the gauge group in topological string theory is complex.

[10] We recall that the representations of $\mathrm{U}(P)$ and S_k are both labeled by a Young tableau.

[11] g encodes the quantum number under SO(2), the rotation group in the two non-compact directions.

wrapping the class[12] $\vec{\Sigma}\cdot\vec{Q} \in H_2(X, L, \mathbf{Z})$ transforming in a representation R of $\mathrm{U}(P)$ and ending on the $D4$-branes wrapping L.

Knowledge of the open GV invariants $\widehat{N}_{Rg\vec{Q}}$ and the holonomy matrix V corresponding to a D-brane configuration in X determines using (2.8) the open topological string partition function in X to all orders in perturbation theory.

3. Open strings in X = closed strings in X_b

We are now going to evaluate the open string partition function in a Calabi-Yau X (2.8) and show that the resulting open+closed partition function in X takes precisely the form of a closed string partition function (2.4) on a new Calabi-Yau manifold X_b! The physical interpretation of X_b is as the Calabi-Yau space obtained by letting the D-branes in X undergo a geometric transition. From the identification of partition functions we can compute the closed GV invariants[13] $n_g^{\vec{Q}_b}(X_b)$ in X_b in terms of the open $\widehat{N}_{Rg\vec{Q}}(X)$ and closed $n_g^{\vec{Q}}(X)$ GV invariants in X.

The open+closed topological string partition function in X has a contribution from the open string sector living on the D-brane configuration under study and one from the closed string sector. Therefore, the partition function factorizes into two pieces

$$Z_{o+c}(X) = Z_o(g_s, t, V)\cdot Z_c(g_s, t), \tag{3.1}$$

the first arising from world-sheets with boundaries while the second one from world-sheets without boundaries. $n_g^{\vec{Q}}(X)$ determines $Z_c(g_s, t)$ while $\widehat{N}_{Rg\vec{Q}}(X)$ together with the holonomy of the gauge field determines $Z_o(g_s, t, V)$. Since our goal is to show that the open+closed partition function in X (3.1) takes the form of a closed string partition function $Z_c(X_b)$, the main task is to show that the open string contribution to (3.1) can be rewritten as a closed string amplitude. Of course, the detailed form of the closed string partition function in X_b will depend on the closed string partition function in X.

The open string partition function on such a D-brane configuration in X is completely characterized by the corresponding open GV invariants in X *and* by specifying the holonomy of the gauge field $\mathcal{A}$ living on the D-brane configuration. Since the D-branes wrap a Lagrangian submanifold L with $b_1(L)\neq 0$, the D-brane amplitude depends on the gauge invariant[14] holonomy matrix

$$V = P\exp\left[-\left(\oint_\beta \mathcal{A} + \int_D \mathcal{J}\right)\right], \tag{3.2}$$

where $\mathcal{J}$ is the complexified Kähler form, $\beta \in H_1(L)$ and D is a two-chain with $\partial D = \beta$. Geometrically, the holonomy of the gauge field (3.2) is gauge equivalent to the "position"[15]

[12] $H_2(X, L)$ denotes the relative homology group.

[13] We note that $\vec{Q}_b \in \mathbf{Z}^{b_2(X_b)}$ while $\vec{Q} \in \mathbf{Z}^{b_2(X)}$. We shall see that if R is parametrized as in figure 1, then $H_2(X_b, \mathbf{Z}) \simeq H_2(X, \mathbf{Z}) \oplus \mathbf{Z}^{2m}$.

[14] This is gauge invariance under closed string field theory gauge transformations, which act by $\mathcal{J}\to \mathcal{J}+d\Lambda,\ \mathcal{A}\to\mathcal{A}-\Lambda$.

[15] The position is defined only up to Hamiltonian deformations, which are gauge symmetries of the A-model open string field theory.

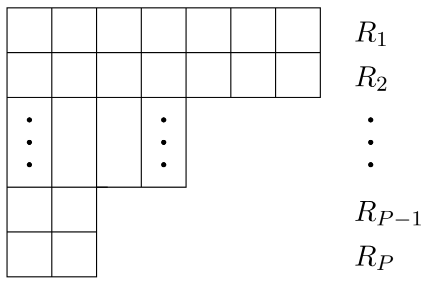

Figure 2: A Young tableau R. R_i is the number of boxes in the i-th row. It satisfies $R_i \geq R_{i+1}$.

of the branes in X. Therefore, the holonomy is part of the data that the open string theory depends on.

Following [4], we turn on discrete values of the holonomy matrix (3.2) determined by a Young tableau R. For a configuration of P D-branes the holonomy matrix can be diagonalized

$$V \equiv U_R = \operatorname{diag}\left(e^{-a_1}, e^{-a_2}, \ldots, e^{-a_P}\right), \tag{3.3}$$

where the eigenvalue a_i corresponds to the "position" of the i-th brane, which is given by [4]

$$a_i \equiv \oint_\beta \mathcal{A}_i + \int_D \mathcal{J} = g_s\left(R_i - i + P + \frac{1}{2}\right), \quad i = 1, \ldots, P. \tag{3.4}$$

R_i is the number of boxes in the i-th row of the Young tableau R:

The explicit formula for the closed GV invariants in X_b depends on the closed GV invariants in X, the open GV invariants of the D-brane configuration in X and on the holonomy of the gauge field (3.3) on the branes, which is determined by a Young tableau R. The most interesting contribution to the formula we derive for the closed GV invariants in X_b arises from the open string partition function of the brane configuration in X, since $Z_c(g_s, t)$ in (3.1) already takes the form of a closed string partition function.

We start by performing our computations for the case when X is the resolved conifold geometry. Apart from already capturing the closed string, bubbling Calabi-Yau interpretation of D-branes in a simple setting, it also has interesting applications to knot invariants. We find that the closed topological string partition function on certain bubbling Calabi-Yau manifolds are invariants of knots in S^3.

We want to compute the open+closed topological string partition function on the resolved conifold geometry. In order to define the open string partition function we must first specify a D-brane configuration in the resolved conifold giving rise to BRST-invariant boundary conditions on the string world-sheet, corresponding to branes wrapping a Lagrangian submanifold. The resolved conifold is an asymptotically conical Calabi-Yau with base $S^2 \times S^3$ and topology $R^4 \times S^2$. One can construct a Lagrangian submanifold L for every knot α in the S^3 at asymptotic infinity [14, 15]. We can then study the open string theory defined by D-branes wrapping these Lagrangian submanifolds, which have topology $L \simeq \mathrm{R}^2 \times S^1$ and end on a knot α at asymptotic infinity.

We consider the open+closed string partition when P D-branes wrap a Lagrangian submanifold L associated to an arbitrary knot $\alpha \subset S^3$. There are several contributions, from both the open and closed string sector.

The closed string contribution is well known [11, 13]:

$$Z_c(g_s,t) = M(q) \cdot \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n\,[n]^2} e^{-nt} \right). \tag{3.5}$$

Comparing with the general formula for the closed string partition function in terms of the closed GV invariants (2.4) one finds that there is a unique non-vanishing closed GV invariant in the resolved conifold geometry, given by $n_0^1 = -1$. For the resolved conifold geometry $b_2(X) = 1$ — and $\chi(X) = 2$ — and $t = \int_{S^2} \mathcal{J}$ parametrizes the complexified size of the S^2.

The open string contribution to the partition function has several pieces. One contribution is captured by the open string partition function in (2.8). The holonomy of the gauge field (3.2) around the non-contractible one-cycle β in the Lagrangian L, — labeled by the knot α[16] — must be given to completely specify the D-brane configuration, and the corresponding open string theory. This is because the holonomy of the gauge field determines the positions of the D-branes up to Hamiltonian deformations[17] [4], which are gauge symmetries of the A-model open string field theory. Following [4] we now turn on a non-trivial holonomy $V = U_R$ (3.2) labeled by a Young tableau R (3.3), (3.4). Turning on a non-trivial holonomy has the effect of separating the branes, and therefore making the off-diagonal open strings massive. Integrating these fields out also contributes to the open string amplitude on the D-brane configuration. Combining the various terms we have that the complete open string partition function is given by

$$Z_o(g_s,t,V=U_R) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \left[-\sum_{1\le i<j\le P} e^{-n(a_i-a_j)} + \sum_{\vec{k}} \frac{1}{z_{\vec{k}}} f_{\vec{k}}(q^n, e^{-nt}) \mathrm{Tr}_{\vec{k}} U_R^n \right] \right) \tag{3.6}$$

where

$$\exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \sum_{1\le i<j\le P} e^{-n(a_i-a_j)} \right) = \prod_{1\le i<j\le P} \left(1 - e^{-(a_i-a_j)}\right) \tag{3.7}$$

arises by integrating out the off-diagonal massive open strings. From a world-sheet perspective this last contribution arises from world-sheet annuli connecting the various D-branes.[18]

[16] Note that the knot $\alpha \subset S^3$ is contractible in L.

[17] A Hamiltonian deformation is generated by a vector v in the normal bundle of L of the form $v^\mu = (w^{-1})^{\mu\nu}\partial_\nu f$ for arbitrary f, where $w_{\mu\nu}$ is the Kähler form of the symplectic manifold X.

[18] Though (3.7) looks like a fermion determinant if we naively apply the argument of [1], the massive open string is a boson. The argument does not really apply because the open string is not localized along an S^1. It instead applies to the related toric situation where an open string stretches between one brane

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By combining the closed string partition function (3.5) with the open string partition function (3.6), we find that the open+closed partition function for a configuration of P D-branes wrapping a Lagrangian submanifold L in the resolved conifold is given by:

$$Z_{o+c} = M(q)\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\left[-\frac{e^{-nt}}{[n]^2}-\sum_{1\le i<j\le P}e^{-n(a_i-a_j)}+\sum_{\vec{k}}\frac{1}{z_{\vec{k}}}f_{\vec{k}}(q^n,e^{-nt})\mathrm{Tr}_{\vec{k}}U_R^n\right]\right). \quad (3.8)$$

The first step in identifying the open+closed string partition function in (3.8) as a purely closed string amplitude is to write the contribution from the off-diagonal massive open strings in (3.7) as a closed string world-sheet instanton expansion. For this purpose, it is convenient to parametrize the Young tableau using the coordinates in figure 1. Then the following useful identity can be derived (see appendix A)

$$\begin{aligned}\xi(q)^P \exp&\left(-\sum_{n=1}^{\infty}\frac{1}{n}\sum_{1\le i<j\le P}e^{-n(a_i-a_j)}\right)\\ &= M(q)^m\exp\left(\sum_{n=1}^{\infty}\frac{1}{n\,[n]^2}\left[\sum_{1\le I\le J\le 2m-1}(-1)^{J-I+1}e^{-n(t_I+t_{I+1}+\ldots+t_J)}\right]\right), \end{aligned}\quad (3.9)$$

where we have identified

$$t_I = g_s l_I \qquad I=1,\ldots,2m \quad (3.10)$$

with l_I being the coordinates of the Young tableau in figure 1. $M(q)$ is the MacMahon function (2.5) and $\xi(q)=\prod_{j=1}^{\infty}(1-q^j)^{-1}$. In this way we have written the contribution from open string world-sheets with annulus topology as a closed string instanton expansion.

We can also derive the following formula for the holonomy of the gauge field on the branes (see appendix A)

$$\mathrm{Tr}_{\vec{k}}U_R^n = \prod_{j=1}^{\infty}\left(\frac{\sum_{I=1}^{m}e^{-njT_{2I-1}}-e^{-njT_{2I}}}{[nj]}\right)^{k_j}, \quad (3.11)$$

with U_R given in (3.3), (3.4). Here

$$T_I = \sum_{J=I}^{2m} t_J \quad (3.12)$$

and $[nj]=q^{nj/2}-q^{-nj/2}$, where $q=e^{-g_s}$. Therefore, the contribution of the holonomy matrix to the open string amplitude (3.6) also takes the form of a world-sheet instanton expansion with Kähler parameters t_I, with $I=1,\ldots,2m$. For later purposes it is convenient to introduce the notation

$$e^{-nT_o}\equiv(e^{-nT_1},e^{-nT_3},\ldots,e^{-nT_{2m-1}}),\quad e^{-nT_e}\equiv(e^{-nT_2},e^{-nT_4},\ldots,e^{-nT_{2m}}). \quad (3.13)$$

D_1 wrapping L_1 and another D_2 wrapping L_2. Here L_1 and L_2 are two Lagrangians that can combine and move off to infinity [16]. The open string is localized along $L_1\cap L_2=S^1$, and the argument of [1] implies that it contributes the bosonic determinant $1/(1-e^{-\Delta a})$. If D_1 and D_2 both wrap L_1 (or L_2), the contribution from the stretched open string is the inverse $(1-e^{-\Delta a})$, which appears in (3.7). We thank M. Aganagic for explaining this to us.

A crucial step in uncovering the closed string interpretation of open string amplitudes in topological string theory is to use the following identity (proven in appendix C using CFT techniques, which are reviewed in appendix B)

$$\sum_{\vec{k}} \frac{1}{z_{\vec{k}}} \chi_{R_1}(C(\vec{k})) \prod_{j=1}^{\infty} \left(\sum_{I=1}^{m} \lambda_I{}^j - \sum_{I=1}^{m} \eta_I{}^j \right)^{k_j} = \sum_{R_1, R_2, R_3} (-1)^{|R_3|} N_{R_2 R_3}^{R_1} s_{R_2}(\lambda) s_{R_3^T}(\eta), \quad (3.14)$$

where $\lambda = (\lambda_I)$ and $\eta = (\eta_I)$ with $I = 1, \ldots, m$ are arbitrary variables. The left hand side of (3.14) enters in the parametrization of the open string partition function in (2.8) by using (2.11). The symbol $N_{R_2 R_3}^{R_1}$ denotes the Littlewood-Richardson coefficients of $\mathrm{U}(P)$, which determine the number of times the representation R_1 of $\mathrm{U}(P)$ appears in the tensor product of representations R_2 and R_3 of $\mathrm{U}(P)$. R_3^T is the representation of $\mathrm{U}(P)$ obtained by transposing the Young tableau R_3. Finally, $s_R(x)$ is a Schur polynomial of $\mathrm{U}(m)$, which is labeled by a Young tableau R. It is defined by taking the trace[19] in the representation R

$$s_R(x) \equiv \mathrm{Tr}_R X, \quad (3.15)$$

where X is an $m \times m$ diagonal matrix with entries $X \equiv \mathrm{diag}(x_1, \ldots, x_m)$.

We can now use (3.11), (3.14)[20] to write the second term in the open string partition function on the resolved conifold (3.6) as follows:

$$\begin{aligned} &\sum_{\vec{k}} \frac{1}{z_{\vec{k}}} f_{\vec{k}}(q^n, e^{-nt}) \mathrm{Tr}_{\vec{k}} U_R^n \\ &\quad = \sum_{g=0}^{\infty} \sum_{Q \in \mathbf{Z}} \sum_{R_1, R_2, R_3} \frac{1}{[n]^{2-2g}} \widehat{N}_{R_1 g Q} (-1)^{|R_3|} N_{R_2 R_3}^{R_1} s_{R_2}(e^{-nT_o}) s_{R_3^T}(e^{-nT_e}) e^{-nQt}. \quad (3.16) \end{aligned}$$

We note that the factor $[nj]^{k_j}$ in the definition of $f_{\vec{k}}$ in (2.11) precisely cancels with an identical factor in (3.14).

Therefore, we have proven that the open+closed partition function on the resolved conifold (3.8) can be written as follows:[21]

$$\begin{aligned} Z_{o+c} = M(q)^{m+1} \exp \Bigg(\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n\ [n]^{2-2g}} \Bigg[&- \delta_{g0} e^{-nt} \\ &+ \delta_{g0} \sum_{1 \le I \le J \le 2m-1} (-1)^{J-I+1} e^{-n(t_I + t_{I+1} + \ldots + t_J)} \\ &+ \sum_{Q \in \mathbf{Z}} \sum_{R_1, R_2, R_3} \widehat{N}_{R_1 g Q} (-1)^{|R_3|} N_{R_2 R_3}^{R_1} s_{R_2}(e^{-nT_o}) s_{R_3^T}(e^{-nT_e}) e^{-nQt} \Bigg] \Bigg). \quad (3.17) \end{aligned}$$

[19] In terms of the fundamental representation, we have that $\mathrm{Tr}_R X = \sum_{\vec{k}} \frac{1}{z_{\vec{k}}} \chi_R(C(\vec{k})) \prod_j (\mathrm{Tr} X^j)^{k_j}$.

[20] For the resolved conifold there is only one Kähler modulus, which we denote by t.

[21] In writing this, we have dropped an ambiguous factor proportional to $\xi(q)$, which does not affect the answer to any order in perturbation theory [17].

A quick glance at the formula for the closed topological string partition function in terms of closed GV invariants (2.4) confirms that the open+closed partition function in the resolved conifold (3.17) takes precisely the form of a closed string partition function on a different Calabi-Yau space X_b. Moreover, by using that the Littlewood-Richardson coefficients $N_{R_2R_3}^{R_1}$ are integers and that a Schur polynomial $s_R(M)$ is a symmetric polynomial of the eigenvalues of M with integer coefficients, we can conclude that the coefficients in (3.17) have the correct integrality properties for a closed string amplitude parametrized by closed GV invariants. Therefore, we have proven that the open+closed string partition function on the resolved conifold takes precisely the form of a closed string partition function in another Calabi-Yau X_b with the correct *integrality* properties!

It follows from the expression in (3.17) that the Calabi-Yau manifold X_b has different topology than the Calabi-Yau space we started with. In fact, by looking at the exponent of $M(q)$ in (3.17) we have shown that $\chi(X_b) = 2m+2$. The appearance of the extra cycles has a simple physical intepretation. The branes in the resolved conifold have undergone a geometric transition and have been replaced by flux. Fluxes in the topological string correspond precisely to non-trivial periods of the complexified Kähler form $\mathcal{J}$. In this picture, the original branes disappear and leave behind a collection of non-contractible cycles on which their flux is supported. It is this picture that warrants the description of X_b as a bubbling Calabi-Yau.

It is now straightforward to extend the computation of the open+closed partition function to an arbitrary Calabi-Yau X. The open+closed partition function of a D-brane configuration in X is given by:

$$\begin{aligned} Z_{o+c} = M(q)^{\frac{\chi(X)+2m}{2}} \exp\Bigg(\sum_{g=0}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n\ [n]^{2-2g}} \Bigg[&\sum_{\vec{Q}} n_g^{\vec{Q}} e^{-n\vec{Q}\cdot\vec{t}} \\ &+ \delta_{g0} \sum_{1\leq I\leq J\leq 2m-1} (-1)^{J-I+1} e^{-n(t_I+t_{I+1}+\ldots+t_J)} \\ &+ \sum_{\vec{Q}} \sum_{R_1,R_2,R_3} \widehat{N}_{R_1 g\vec{Q}} (-1)^{|R_3|} N_{R_2R_3}^{R_1} s_{R_2}(e^{-nT_o}) s_{R_3^T}(e^{-nT_e}) e^{-n\vec{Q}\cdot\vec{t}} \Bigg]\Bigg). \end{aligned} \tag{3.18}$$

The integers $n_g^{\vec{Q}}$ are the closed GV invariants in X, which determine the closed string partition function in X, where now $\vec{Q}\in \mathbf{Z}^{b_2(X)}$. As before, the integers $\widehat{N}_{R' g\vec{Q}}$ are the open GV invariants of the D-brane configuration in X. Just as in the case when X is the resolved conifold, the open+closed partition function (3.18) takes precisely the form of a closed string partition function in X_b (2.4), with integral closed GV invariants. This explicitly shows that the physics of D-branes in X can be either described by open+closed string theory in X or equivalently by closed string theory on a topologically different manifold X_b. Showing that the open+closed string theory in X has a closed string interpretation in X_b does not rely on explicitly knowing the open and closed GV invariants in X. Nevertheless, since the open and closed partition function take a very particular form in topological string theory — being parametrized by integer invariants –, we can show that we the open string amplitude in X takes the form of a closed string amplitude in X_b.

We can explicitly compute the closed GV invariants $n_g^{\vec{Q}_b}(X_b)$ in X_b in terms of the open $\widehat{N}_{Rg\vec{Q}}$ and closed $n_g^{\vec{Q}}$ GV invariants in X by comparing the open+closed string partition function in X (3.18) with the general expression for the closed string partition function in topological string theory (2.4). By matching the two series we get:

$$\sum_{\vec{Q}_b} n_g^{\vec{Q}_b}(X_b) e^{-\vec{Q}_b\cdot\vec{t}} = \sum_{\vec{Q}} n_g^{\vec{Q}} e^{-\vec{Q}\cdot\vec{t}} + \delta_{g0} \sum_{1\le I\le J\le 2m-1} (-1)^{J-I+1} e^{-t_I - t_{I+1} - \ldots - t_J} + \sum_{\vec{Q}} \sum_{R_1 R_2 R_3} \widehat{N}_{R_1 g\vec{Q}} e^{-\vec{Q}\cdot\vec{t}} (-1)^{|R_3|} N^{R_1}_{R_2 R_3} s_{R_2}(e^{-T_o}) s_{R_3^T}(e^{-T_e}). \quad (3.19)$$

By comparing the two series one can explicitly calculate $n_g^{\vec{Q}_b}(X_b)$ in terms of $\widehat{N}_{R_1 g\vec{Q}}$ and $n_g^{\vec{Q}}$. In appendix D, we rewrite (3.19) in a form in which it is easy to obtain the closed GV invariants in X_b from the open and closed GV invariants in X.

Continuous v.s. discrete holonomies and framing dependence. Holonomy taking discrete values plays a crucial role in the discussion in [4] and this paper. On the other hand, most topological string literature starting with [1] has assumed that holonomy takes continuous values. It is natural to ask what is the relation between the two pictures.

Our proposal is that the partition function in one picture with one framing is a linear combination of partition functions in the other picture with an appropriate framing. We now explain this statement in some detail. Let us assume that the Lagrangian submanifold L the D-branes wrap has topology of $\mathrm{R}^2 \times S^1$, which can be regarded as solid torus. At asymptotic infinity, the geometry is a cone over T^2. Given L, there is a unique one-cycle of T^2 that is contractible in L. In fact, as one moves from one point to another one in the quantum moduli space of such D-branes, the original contractible cycle can become non-contractible while another cycle becomes contractible. In other words, the quantum moduli space contains topologically distinct Lagrangian submanifolds that are related by a flop. The open string partition function $Z_o(g_s, V; f_1)$ is a wave function in Chern-Simons theory on the T^2 at infinity. The definition of the wave function involves framing(=the choice of polarization) f_1, i.e., the choice of variables corresponding to a coordinate and its conjugate momentum. In the case of Chern-Simons theory on T^2, polarization is fixed by choosing a pair of symplectic generators (α, β) such that $\#(\alpha\cap\beta) = 1$. $\oint_\alpha \mathcal{A}$ plays the role of a coordinate and $\oint_\beta \mathcal{A}$ the role of the conjugate momentum. g_s plays the role of the Planck constant [18]. The conventional picture of holonomy is such that $V \sim \exp - \oint_\alpha \mathcal{A}$, where α is a non-contractible cycle. Since $\oint_\alpha \mathcal{A}$ is a periodic variable, the conjugate momentum $\oint_\beta \mathcal{A}$ gets quantized in units of g_s. A basis state $|R\rangle$ of the Hilbert space in our polarization is labeled by a Young tableau R, and this state corresponds to $\exp - \oint_\beta \mathcal{A} = U_R$ [18]. On the other hand, the state in which $\exp - \oint_\alpha \mathcal{A}$ equals V is $|V\rangle = \sum_R \mathrm{Tr}_R V |R\rangle$. We expect that there is a point in the moduli space where α is a non-contractible cycle of L. We also expect that the two open string partition functions are related as $Z_o(g_s, V; f_1) = \sum_R \mathrm{Tr}_R V Z_o(g_s, U_R; f_2)$ with appropriate framing f_2. This is indeed what happens for the D-branes corresponding to unknot in S^3 up to normalization and a shift in the Kähler modulus [8].

Knot invariants from closed strings in bubbling Calabi-Yau manifolds. In [4] we identified the D-brane configurations[22] in the resolved conifold X corresponding to a Wilson loop in $\mathrm{U}(N)$ Chern-Simons theory on S^3. The brane configuration depends on the knot $\alpha \subset S^3$ and on the choice of a representation R of $\mathrm{U}(N)$, which is the data on which the Wilson loop depends on (see [4] for the details of the brane configuration).

This identification was explicitly verified for the case when α is the unknot and for an arbitrary representation R. In addition, we noticed that the D-brane configuration[23] in the resolved conifold corresponding to the unknot and for arbitrary representation R, shown in figure 3(a), could be given a purely closed string interpretation in terms of the closed string partition function on a bubbling Calabi-Yau X_b of ladder type, shown in figure 3(b). More concretely, we showed that [4]

$$\left\langle \mathrm{Tr}_R P \exp - \oint_\alpha A \right\rangle = Z_{o+c}(X) = Z_c(X_b), \tag{3.20}$$

where

$$Z_{o+c}(X) = M(q) \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \left[-\frac{e^{-nt}}{[n]^2} - \sum_{i<j} e^{-n(a_i - a_j)} + \sum_{i=1}^{P} \frac{e^{-na_i} - e^{-n(t+a_i)}}{[n]} \right] \right) \tag{3.21}$$

is the open+closed string partition function in the resolved conifold X, and

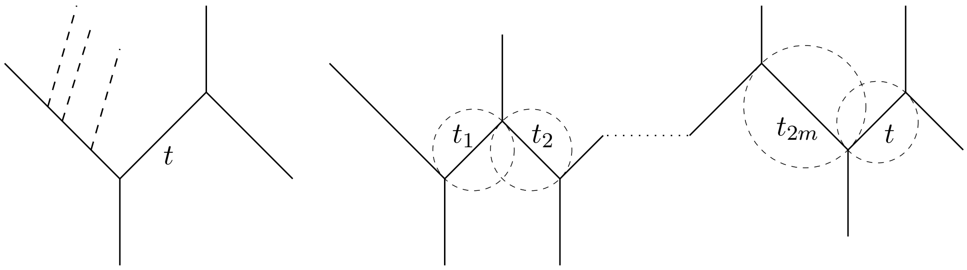
$$\begin{aligned} Z_c(X) = M(q)^{m+1} \exp \sum_{n=1}^{\infty} \frac{1}{n[n]^2} \Bigg(& - \sum_{1 \le I \le 2m+1} e^{-nt_I} + \sum_{1 \le I \le 2m} e^{-n(t_I + t_{I+1})} \\ & - \sum_{1 \le I \le 2m-1} e^{-n(t_I + t_{I+1} + t_{I+2})} \ldots - e^{-n(t_1 + \ldots + t_{2m+1})} \Bigg) \end{aligned} \tag{3.22}$$

is the closed string partition function in X_b with $t_{2m+1} \equiv t$. The equality $Z_{o+c}(X) = Z_c(X_b)$ is of course the special case of the result in the present paper. By comparing (3.21) with (3.17), we see that $\widehat{N}_{\Box, g=0, Q=0} = 1$ and $\widehat{N}_{\Box, g=0, Q=1} = -1$ are the only non-zero open GV invariants. It can be seen that (3.22) agrees with (3.17). One consequence of this identification is that closed topological string theory on bubbling Calabi-Yau manifolds X_b yield knot invariants for the unknot.

In this paper we have shown that *any* brane configuration in a Calabi-Yau manifold — so in particular in the resolved conifold — has a purely closed string interpretation. Since we know [4] which brane configuration corresponds to a Wilson loop for arbitrary knot α and representation R, we can associate to the bubbling Calabi-Yau obtained from this brane configuration a knot. This set of connections uncovers an interesting relation between closed GV invariants in bubbling Calabi-Yau manifolds X_b and invariants of knots in S^3. It implies that the closed string partition function on appropriate bubbling Calabi-Yau manifolds X_b are invariants of knots on S^3.

[22] The convention for the distinction of brane/anti-brane here is the opposite of [4].

[23] As explained in [4], a given Wilson loop can be represented either in terms of D-branes or anti-branes in the resolved conifold, in an analogous fashion to the AdS description of half-BPS Wilson loops [19]. Both brane configurations give rise to the same bubbling Calabi-Yau X_b.



(a) (b)

Figure 3: (a) The resolved conifold and D-branes with holonomy U_R inserted on an outer edge. (b) The bubbling Calabi-Yau X_b after geometric transition of the D-branes. The Kähler moduli are given by $t_I = g_s l_I$, $I = 1, \ldots, 2m$, where l_I are defined in figure 1.

4. Geometric transitions in toric Calabi-Yau's

In this section we study the geometric transitions giving rise to bubbling Calabi-Yau manifolds in the set-up of toric Calabi-Yau manifolds. In addition to the general picture of geometric transitions presented in the previous section, here we are able to concretely identify both the D-brane configurations and the bubbling Calabi-Yau manifolds. We explain how these geometric transitions can be understood by a combination of complex structure deformation and a local version of conifold transition. Furthermore we explicitly show, by using the topological vertex techniques, that the open string partition function in a given D-brane configuration is precisely the closed string partition function in the corresponding bubbling Calabi-Yau.

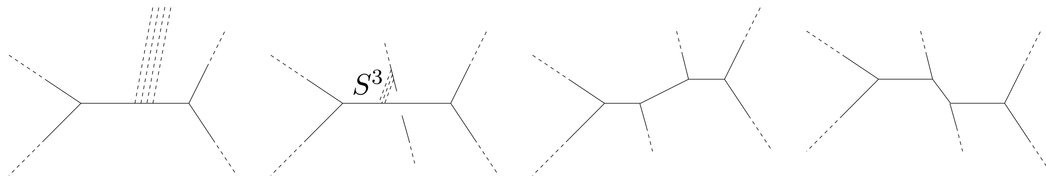
4.1 Local Gopakumar-Vafa duality

Take an arbitrary toric Calabi-Yau manifold specified by a toric diagram. Let us focus on one of the edges. Without losing generality we assume that it is an internal edge.[24] Consider m non-compact branes wrapping a Lagrangian submanifold as shown in figure 4(a). The submanifold has the topology of $\mathrm{R}^2 \times S^1$, and preserves an $\mathrm{U}(1)^2 \subset \mathrm{U}(1)^3$ symmetry. As explained in [20], it is possible to modify the geometry so that the new geometry has a compact 3-cycle of S^3 topology in the edge.[25] Near the S^3 the local geometry is that of the deformed conifold. The new geometry is not toric, but has the structure of an $\mathrm{R} \times T^2$ fibration [21]. By a complex structure deformation that makes the S^3 infinitely large, one recovers the original toric Calabi-Yau. The A-model amplitude is invariant under the complex structure deformation.

The m branes now wrap the S^3 as shown in figure 4(b). In the limit of infinite S^3 size we get m non-compact D-branes ending on the edge in the original geometry, see figure 4(a). In the original geometry, the non-compact Lagrangian submanifold has topology of $\mathrm{R}^2 \times S^1$, which we regard as a solid torus. In particular it has a non-contractible S^1 cycle. The non-compact Lagrangian is compactified to S^3 in the modified geometry. If we focus on the

[24] By making the internal edge infinitely long one can trivially make it external.

[25] In fact there is an infinite family of such modifications labeled by an integer p. p specifies framing of the non-compact branes as well as the orientation of the new line in 4(b).



(a) (b) (c) (d)

Figure 4: (a) Non-compact D-branes (dashed lines ending on edges) in a toric Calabi-Yau manifold. The framing of the branes is specified by a vector f. (b) The geometry can be modified without changing the amplitude while making the brane world-volume a compact S^3. (c) The compact branes get replaced by a new 2-cycle upon geometric transition. (d) Geometric transition of anti-branes produces a flopped geometry.

Lagrangian alone, compactification is achieved by gluing another copy of the solid torus to the first copy after applying the $S \in \mathrm{SL}(2, \mathbf{Z})$ transformation on the T^2 boundary. The non-contractible S^1 becomes contractible in the new copy. The Chern-Simons path integral on the new copy of the solid torus prepares a state on T^2, which is the ground state because we insert no Wilson loop. After the S transformation, the ground state induces certain holonomy along the S^1 proportional to the Weyl vector of $\mathrm{U}(m)$ [18]:

$$- \oint \mathcal{A} = \mathrm{diag} \left(g_s \left[-i + \frac{1}{2} + \frac{m}{2} \right] \right)_{i=1}^{m} . \tag{4.1}$$

We now apply the local Gopakumar-Vafa duality [22] to the branes wrapping the S^3. The m branes disappear and get replaced by a 2-cycle of topology S^2 with complexified Kähler modulus $g_s m$. The local geometry is that of the resolved conifold with Kähler parameter $g_s m$. See figure (c). This makes clear that we need discrete values of holonomy on the branes to have geometric transition.[26]

If replace the branes by anti-branes we obtain a flopped geometry (figure 4(d)).

4.2 Geometric transition of branes in toric Calabi-Yau's

We now verify our proposal for the geometric transition described above. This is done by showing, using the topological vertex formalism [20], that non-compact branes and anti-branes with certain discrete values of holonomy can be replaced by geometries. As in much of recent literature we redefine $q \to q^{-1}$ relative to [20].[27] Basic facts about the topological vertex are summarized in appendix E.

Let us consider an arbitrary toric Calabi-Yau manifold that contains an interior edge as shown in figure 5(a). Without D-branes the part of the partition function corresponding to this edge would be:

$$\sum_R C_{R_1 R_2 R} (-1)^{(n+1)|R|} q^{\frac{1}{2} n \kappa_R} e^{-|R| t} C_{R^T R_3 R_4} . \tag{4.2}$$

[26] Branes with continuous values of the holonomy on an edge are a superposition (integral transform) of branes with discrete values of the holonomy ending on another edge [4]. The integral transform accounts for the change of polarization of Chern-Simons theory on T^2.

[27] This is to ensure that infinite power series that appears in amplitudes involve positive powers of q. Such convention is more natural in relation to the quantum foam picture [23, 24].

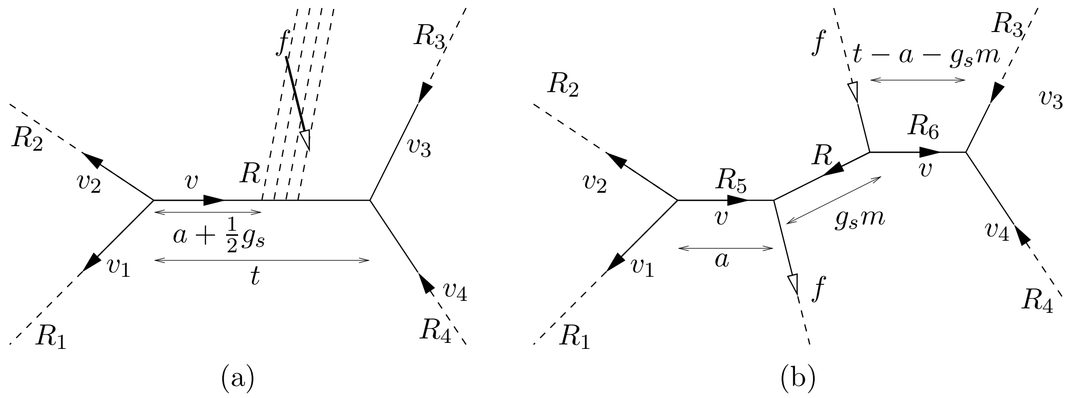


Figure 5: (a) An internal edge of length t in a toric web diagram. $v, v_1, \ldots, v_4$ are the vectors whose components are two coprime integers, and they specify the orientations of the associated edges. They satisfy the conditions $v_1 \wedge v = v_2 \wedge v_1 = v \wedge v_2 = 1 = v_3 \wedge v = v_4 \wedge v_3 = v \wedge v_4$, $v + v_1 + v_2 = 0 = v + v_3 + v_4$. $n := v_1 \wedge v_3$ is the relative framing of the two vertices. We insert m non-compact branes at the positions specified in the figure. f is another vector that specifies the framing of the branes, and satisfies the condition $f \wedge v = 1$. The integer $p := f \wedge v_1$ enters the gluing rule of vertices. (b) After the geometric transition the branes get replaced by a new S^2 represented by the edge of length $g_s m$. The orientation of the new external edges is precisely given by the framing vector of the branes.

t is the length of the edge, and n is the relative framing of the two vertices. $C_{R_1R_2R_3}$ is the basic object underlying the topological vertex [20]. $\kappa_R = |R| + \sum_i R_i^2 - 2iR_i$, where R_i is the number of boxes on the i-th row and $|R|$ is the total number of boxes in the Young tableau R. See appendix F for the explicit expression for $C_{R_1R_2R_3}$.

If we insert D-branes[28] with holonomy matrix V in the internal edge, (4.2) is replaced by:

$$\sum_{R,Q_L,Q_R} C_{R_1,R_2,R\otimes Q_L}(-1)^s q^{-F} e^{-L} C_{R^T\otimes Q_R,R_3,R_4} \mathrm{Tr}_{Q_L} V \mathrm{Tr}_{Q_R} V^{-1}. \tag{4.3}$$

If the framing of the branes relative to the left vertex is p then:[29]

$$s = |R| + p(|R| + |Q_L|) + (n+p)(|R| + |Q_R|), \tag{4.4}$$

$$F = \frac{1}{2}p\kappa_{R\otimes Q_L} + \frac{1}{2}(n+p)\kappa_{R^T\otimes Q_R}, \quad L = |R|t + |Q_L|a + |Q_R|(t-a). \tag{4.5}$$

Alternatively we can write (4.3) as:

$$\begin{aligned}&\sum_{R_5,R_6} C_{R_1R_2R_5} \times (-1)^{p|R_5|} q^{-\frac{1}{2}p\kappa_{R_5}} e^{-|R_5|a} \left(\sum_R \mathrm{Tr}_{R_5/R} V (-1)^{|R|} \mathrm{Tr}_{R_6/R^T} V^{-1}\right) \\ &\times (-1)^{(n+p)|R_6|} q^{-\frac{1}{2}(n+p)\kappa_{R_6}} e^{-|R_6|(t-a)} C_{R_6R_3R_4}.\end{aligned} \tag{4.6}$$

[28]In the present convention, a brane here is an anti-brane in [4] and vice versa. This can be confirmed by computing a brane amplitude in the resolved conifold.

[29]Here $a = \int_D \mathcal{J}$ is the complexified area of a holomorphic disk, and $e^{-a}V$ is the gauge invariant open string modulus.

Here $\mathrm{Tr}_{R/R'}(V) := \sum_{R''} N^R_{R'R''}\mathrm{Tr}_{R''}V$ with $N^R_{R'R''}$ being tensor product coefficients.

In appendix F we show that by substituting[30]

$$V = U_m := \mathrm{diag}(q^{m-i+1/2})_{i=1}^m, \tag{4.7}$$

that the expression in the brackets in (4.6), multiplied by[31] $\xi(q)^m \prod_{1\le i<j\le m}(1-q^{j-i})$, is related to the topological vertex:

$$\begin{aligned}\xi(q)^m \prod_{1\le i<j\le m}(1-q^{j-i}) \sum_R \mathrm{Tr}_{R_5/R}U_m(-1)^{|R|}\mathrm{Tr}_{R_6/R^T}U_m^{-1} \\ = M(q)q^{-m|R_6|}q^{-\frac{1}{2}\kappa_{R_5}-\frac{1}{2}\kappa_{R_6}}\sum_R C_{\cdot R_5^T R}(-1)^{|R|}e^{-|R|g_s m}C_{R^T\cdot R_6^T}\end{aligned} \tag{4.8}$$

The expression (4.6) then becomes

$$\begin{aligned}M(q)\sum_{R,R_5,R_6} C_{R_1R_2R_5}(-1)^{p|R_5|}q^{-\frac{1}{2}(p+1)\kappa_{R_5}}e^{-|R_5|a}C_{\cdot R_5^T R}(-1)^{|R|}e^{-|R|g_s m} \\ \times C_{R^T\cdot R_6^T}(-1)^{(n+p+1)|R_6|}q^{-\frac{1}{2}(n+p)\kappa_{R_6}}e^{-|R_6|(t-a-g_s m)}C_{R_6R_3R_4}.\end{aligned} \tag{4.9}$$

This is precisely the contribution from a part of the new geometry shown in figure 5(b), where the branes are replaced by a new S^2! The orientations of the new edges are determined by the framing p of the branes.[32]

Anti-branes. We now demonstrate the geometric transition for anti-branes. Replacing branes by anti-branes is equivalent to the replacement $\mathrm{Tr}_R V \to (-1)^{|R|}\mathrm{Tr}_{R^T}V$ [20]. Since $N^{R_1}_{R_2R_3} = N^{R_1^T}_{R_2^TR_3^T}$,[33] this is equivalent to replacing the bracket in (4.6) by $(-1)^{|R_5|+|R_6|}\sum_R \mathrm{Tr}_{R_5^T/R}V\ (-1)^{|R|}\mathrm{Tr}_{R_6^T/R^T}V^{-1}$. Thus when anti-branes with holonomy V are inserted, the contribution from the part of geometry in figure 5(a) is:

$$\begin{aligned}\sum_{R_5,R_6} C_{R_1R_2R_5}\times(-1)^{(p+1)|R_5|}q^{-\frac{1}{2}p\kappa_{R_5}}e^{-|R_5|a}\left(\sum_R \mathrm{Tr}_{R_5^T/R}V(-1)^{|R|}\mathrm{Tr}_{R_6^T/R^T}V^{-1}\right) \\ \times(-1)^{(n+p+1)|R_6|}q^{-\frac{1}{2}(n+p)\kappa_{R_6}}e^{-|R_6|(t-a)}C_{R_6R_3R_4}.\end{aligned} \tag{4.10}$$

Using the property that $C_{R_1R_2R_3} = q^{-\frac{1}{2}\kappa_{R_1}-\frac{1}{2}\kappa_{R_2}-\frac{1}{2}\kappa_{R_3}}C_{R_3^TR_2^TR_1^T}$ [20] we obtain from (4.8) the relation:

$$\begin{aligned}\xi(q)^m \prod_{1\le i<j\le m}(1-q^{j-i})(-1)^{|R_5|+|R_6|}\sum_R \mathrm{Tr}_{R_5^T/R}U_m(-1)^{|R|}\mathrm{Tr}_{R_6^T/R^T}U_m^{-1} \\ = M(q)q^{-m|R_6|}\sum_R C_{R_5^T\cdot R^T}(-1)^{|R|}e^{-|R|g_s m}C_{\cdot RR_6^T}.\end{aligned} \tag{4.11}$$

[30] The exponent of U_m differs from (4.1) by an i-independent shift that was absorbed in a.

[31] As we saw in section 3, it is natural to include these factors when considering branes with discrete values of the holonomy. The product arises from annuli connecting the branes.

[32] The equality of certain open and closed string amplitudes observed in section 3 of [25] is an example of the geometric transition discussed here. We thank M. Marinõ for pointing this out.

[33] This relation holds for $\mathrm{U}(N)$ in the limit $N \to \infty$, and can be proven, for example, by using (B.8) and (B.11).

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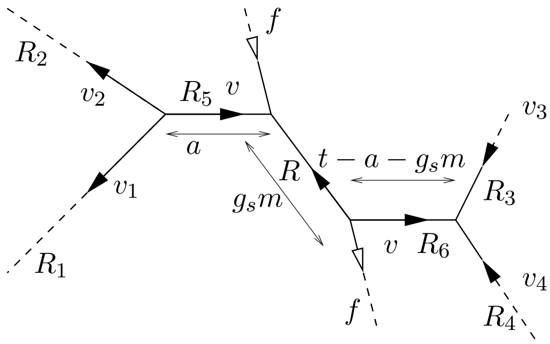

Figure 6: The geometry that is obtained from figure 5(a) through geometric transition of anti-branes. It is related to figure 5(b) by flop.

When combined with formula (4.11), the amplitude (4.10) represents the contribution from the part of the toric geometry shown in figure 6. This is related to the geometry in figure 5(b) by a flop. Again the orientations of the new edges are determined by the framing vector f of the anti-branes.

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A. From open strings to closed strings

In this appendix we give a derivation of formula (3.9):

$$\xi(q)^P \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\sum_{1\le i<j\le P} e^{-n(a_i-a_j)}\right)$$
$$= M(q)^m \exp\left(\frac{1}{n\,[n]^2}\left[\sum_{1\le I\le J\le 2m-1}(-1)^{J-I+1}e^{-n(t_I+t_{I+1}+\ldots+t_J)}\right]\right). \qquad \text{(A.1)}$$

Using the value of the holonomies

$$a_i = g_s\left(R_i - i + P + \frac{1}{2}\right), \qquad i=1,\ldots,P \qquad \text{(A.2)}$$

the exponent on the left hand side of (A.1) can be written as

$$S \equiv \sum_{n=1}^{\infty} \frac{1}{n} \sum_{1 \le i < j \le P} e^{-n g_s (R_i - R_j)} e^{-n g_s (j-i)}. \tag{A.3}$$

We now perform the sum by using the parametrization of the Young tableau in figure 1. There are two classes of contributions. The first class arises when (i, j) belong to the same "block" in the Young tableau, so that $R_i = R_j$, while the second class arises when (i, j) are in different "blocks" and $R_i \neq R_j$. The contribution from rows in the same "block" is given by

$$S_1 = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{I=1}^{m} \sum_{\alpha_I \le i < j \le \beta_I} e^{-n g_s (j-i)}, \tag{A.4}$$

while the contribution from rows in different "blocks" is

$$S_2 = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{1 \le I < J \le m} e^{-n g_s \sum_{a=I}^{J-1} l_{2a}} \sum_{i=\alpha_I}^{\beta_I} e^{n g_s i} \sum_{j=\alpha_J}^{\beta_J} e^{-n g_s j}, \tag{A.5}$$

where

$$\alpha_I = \sum_{a=1}^{I} l_{2a-3} + 1 \qquad \text{and} \qquad \beta_I = \sum_{a=1}^{I} l_{2a-1}, \tag{A.6}$$

and l_i $i = 1, \dots 2m{+}1$ are the coordinates of the Young tableau in figure 1. In writing (A.5) we have used that the number of boxes in the I-th "block" is given by:

$$R_i = \sum_{a=I}^{m} l_{2a} \qquad i \in I\text{-th "block"}. \tag{A.7}$$

The sum in (A.4) can be performed by grouping terms with the same value of $j - i$ and multiplying by the degeneracy; this yields:

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{I=1}^{m} \sum_{k=1}^{\beta_I - \alpha_I} (\beta_I - \alpha_I + 1 - k) e^{-n g_s k}. \tag{A.8}$$

Using the formula

$$\sum_{k=1}^{c-1} (c-k) q^{nk} = -\frac{1}{[n]^2} \left[1 - c(q^n - 1) - q^{nc}\right] \tag{A.9}$$

we get that

$$S_1 = -m \sum_{n=1}^{\infty} \frac{1}{n\,[n]^2} + P \sum_{n=1}^{\infty} \frac{1}{n\,[n]} q^{n/2} + \sum_{I=1}^{m} \sum_{n=1}^{\infty} \frac{1}{n\,[n]^2} e^{-n t_{2I-1}}, \tag{A.10}$$

where $P = \sum_{I=1}^{m} l_{2I-1}$ is the number of rows in the Young tableau and $t_I = g_s l_I$.

The contribution from rows in different blocks can be straightforwardly computed using

$$\sum_{i=1+a}^{b} x^i = \frac{x}{1-x}(x^a - x^b). \tag{A.11}$$

It is given by:

$$S_2 = \sum_{n=1}^{\infty} \frac{1}{n\,[n]^2} \sum_{1\le I<J\le m} \left[e^{-n\sum_{a=2I}^{2J-2} t_a} + e^{-n\sum_{a=2I-1}^{2J-1} t_a} - e^{-n\sum_{a=2I-1}^{2J-2} t_a} - e^{-n\sum_{a=2I}^{2J-1} t_a} \right] \tag{A.12}$$

Therefore, combining (A.10) and (A.12) we get that

$$\xi(q)^P \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \sum_{1\le i<j\le P} e^{-n(a_i - a_j)} \right) = M(q)^m \exp\left(\sum_{n=1}^{\infty} -\frac{1}{n\,[n]^2} \left[\sum_{I=1}^{m} e^{-nt_{2I-1}} \right.\right.$$
$$\left.\left. + \sum_{1\le I<J\le m} \left[e^{-n\sum_{a=2I}^{2J-2} t_a} + e^{-n\sum_{a=2I-1}^{2J-1} t_a} - e^{-n\sum_{a=2I-1}^{2J-2} t_a} - e^{-n\sum_{a=2I}^{2J-1} t_a} \right] \right] \right), \tag{A.13}$$

where

$$M(q) \equiv \exp\left(\sum_{n=1}^{\infty} \frac{1}{n\,[n]^2} \right) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n},$$
$$\xi(q) \equiv \exp\left(\sum_{n=1}^{\infty} \frac{1}{n\,[n]^2} q^{n/2} \right) = \prod_{j=1}^{\infty} (1-q^j)^{-1}. \tag{A.14}$$

The desired formula (A.1) then follows by combining the terms in (A.13).

Likewise, formula (3.11)

$$\mathrm{Tr}_{\vec{k}} U_R^n = \prod_{j=1}^{\infty} \left(\frac{\sum_{I=1}^{m} e^{-njT_{2I-1}} - e^{-njT_{2I}}}{[nj]} \right)^{k_j} \tag{A.15}$$

can be also be derived by splitting the sum over rows into blocks

$$\mathrm{Tr}_{\vec{k}} U_R^n = \prod_{j=1}^{\infty} \left(\sum_{i=1}^{P} e^{-njg_s(R_i - i + P + 1/2)} \right)^{k_j}$$
$$= \prod_{j=1}^{\infty} \left(\sum_{I=1}^{m} e^{-njg_s(\sum_{a=I}^{m} l_{2a} + \sum_{J=1}^{m} l_{2J-1})} e^{-njg_s/2} \sum_{i=\alpha_I}^{\beta_I} e^{njg_s i} \right)^{k_j}, \tag{A.16}$$

where we have used that $P = \sum_{J=1}^{m} l_{2J-1}$. Now we can perform the sums to arrive at the right hand side of (A.15) by using that $T_I = \sum_{i=I}^{2m} g_s l_i$.

B. Operator formalism

In order to derive some of the group theory identities in the paper it is very convenient to exploit the relation between the representation theory of $\mathrm{U}(N)$ and two dimensional bosons and fermions in two dimensions.

Let us consider the mode expansion of a chiral boson $\phi(z)$ and fermions $\psi(z), \overline{\psi}(z)$ in two dimensions, which are related by bosonization/fermionization:

$$\phi(z) \;=\; i\sum_{n\neq 0}\frac{\alpha_n}{nz^n}, \tag{B.1}$$

$$\psi(z) \;=\; \sum_{r\in\mathbf{Z}+\frac{1}{2}}\frac{\psi_r}{z^{r+1/2}}, \overline{\psi}(z) = \sum_{r\in\mathbf{Z}+\frac{1}{2}}\frac{\overline{\psi}_r}{z^{r+1/2}}, \tag{B.2}$$

$$i\partial\phi =: \psi\overline{\psi} :, \quad \psi =: e^{i\phi} :, \quad \overline{\psi} =: e^{-i\phi} : . \tag{B.3}$$

The oscillator modes satisfy the familiar commutation relations:

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}, \qquad \{\psi_r, \overline{\psi}_s\} = \delta_{r+s,0}. \tag{B.4}$$

We can also define a charge conjugation operator C. Charge conjugation C exchanges ψ and $\overline{\psi}$:

$$C\psi(z)C = \overline{\psi}(z),\; C^2 = 1,\; C|0\rangle = |0\rangle. \tag{B.5}$$

Then C acts on $i\partial\phi(z) =: \psi(z)\overline{\psi}(z) :$ as:

$$C\partial\phi(z)C = -\partial\phi(z). \tag{B.6}$$

The connection between Young tableau R and fermions arises from the identification

$$|R\rangle = \prod_{i=1}^{d}\psi_{-a_i-1/2}\overline{\psi}_{-b_i-1/2}|0\rangle, \tag{B.7}$$

where $a_i \equiv R_i - i$, $b_i = R_i^T - i$ are the Frobenius coodinates of R and d is the number of boxes in the diagonal of the Young tableau R.

It follows that:

$$C|R\rangle = (-1)^{|R|}|R^T\rangle. \tag{B.8}$$

Let us now define [23] the operator

$$\Gamma_\pm(z) := \exp\sum_{n=1}^{\infty}\frac{z^{\pm n}}{n}\alpha_{\pm n}, \tag{B.9}$$

which satisfies

$$\Gamma_+(z_+)\Gamma_-(z_-) = \frac{1}{1-z_+/z_-}\Gamma_-(z_-)\Gamma_+(z_+), \quad \Gamma_+(z)|0\rangle = |0\rangle,\; \langle 0|\Gamma_-(z) = \langle 0|. \tag{B.10}$$

The skew Schur functions can be conveniently expressed as

$$s_{R/Q}(x) = \langle R|\prod_i \Gamma_-(x_i^{-1})|Q\rangle = \langle Q|\prod_i \Gamma_+(x_i)|R\rangle. \tag{B.11}$$

The familiar Schur polynomials $s_R(x)$ arise when $|Q\rangle = |0\rangle$. In terms of these the skew Schur polynomials are given by

$$s_{R/Q}(x) = \sum_{R'} N^R_{QR'} s_{R'}(x), \tag{B.12}$$

where $N^R_{QR'}$ are the Littlewood-Richardson coefficients.

The following formula will come in handy in appendix F

$$e^{sL_0}\Gamma_\pm(z)e^{-sL_0} = \Gamma_\pm(e^{-s}z), \tag{B.13}$$

where

$$L_0 = \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n. \tag{B.14}$$

C. An identity for integrality

Let us prove the equation (3.14).

$$\sum_{\vec{k}} \frac{1}{z_{\vec{k}}}\chi_{R_1}(C(\vec{k}))\prod_{j=1}^{\infty}\left(\sum_{I=1}^{m}{\lambda_I}^j - \sum_{I=1}^{m}{\eta_I}^j\right)^{k_j} = \sum_{R_1,R_2,R_3}(-1)^{|R_3|}N^{R_1}_{R_2R_3}s_{R_2}(\lambda)s_{R_3^T}(\eta), \tag{C.1}$$

This is a generalization of (7.29) in [21]. It was used there for a similar purpose, and was originally derived in [2].

Proof. Consider oscillators α_n for a chiral boson as in (B.3). Let us consider the state

$$\begin{aligned}
|\lambda,\eta\rangle &\equiv \sum_{\vec{k}} \frac{1}{z_{\vec{k}}}\prod_{j=1}^{\infty}\left(\sum_{I=1}^{m}{\lambda_I}^j - \sum_{i=I}^{m}{\eta_I}^j\right)^{k_j}\prod_{j=1}^{\infty}\alpha_{-j}^{k_j}|0\rangle \\
&= \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{I=1}^{m}{\lambda_I}^n - \sum_{I=1}^{m}{\eta_I}^n\right)\alpha_{-n}\right)|0\rangle \\
&= \prod_{I=1}^{m}\Gamma_-(\lambda_I^{-1})\prod_{I=1}^{m}\Gamma_-^{-1}(\eta_I^{-1})|0\rangle.
\end{aligned} \tag{C.2}$$

The left hand side of (C.1) is $\langle R_1|\lambda,\eta\rangle$, where $\langle R_1|$ is the fermionic Fock state associated with R_1 in (B.7).

It can also be evaluated as follows:

$$\begin{aligned}
\langle R_1|\lambda,\eta\rangle &= \sum_{R_2}\langle R_1|\prod_I\Gamma_-(\lambda_I^{-1})|R_2\rangle\langle R_2|\prod_I\Gamma_-^{-1}(\eta_I^{-1})|0\rangle \\
&= \sum_{R_2}\langle R_1|\prod_I\Gamma_-(\lambda_I^{-1})|R_2\rangle(-1)^{|R_2|}\langle R_2^T|C\prod_I\Gamma_-^{-1}(\eta_I^{-1})|0\rangle \\
&= \sum_{R_2}(-1)^{|R_2|}s_{R_1/R_2}(\lambda)s_{R_2^T}(\eta) \\
&= \sum_{R_2,R_3}(-1)^{|R_2|}N^{R_1}_{R_2R_3}s_{R_3}(\lambda)s_{R_2^T}(\eta).
\end{aligned} \tag{C.3}$$

where we have used (B.11). This proves (C.1) after relabeling R_2 and R_3.

D. From closed strings to open strings

On the right hand side of (3.19), let us focus on:

$$\sum_{R_1 R_2 R_3} \widehat{N}_{R_1 g\vec{Q}} (-1)^{|R_3|} N_{R_2 R_3}^{R_1} s_{R_2}(e^{-T_o}) s_{R_3^T}(e^{-T_e}). \tag{D.1}$$

In the operator formalism, we can write this as:

$$\sum_{R_1} \widehat{N}_{R_1 g\vec{Q}} \langle R_1| \prod_i \Gamma_-(e^{T_{o,i}}) \prod_i \Gamma_-(e^{T_{e,i}})^{-1}|0\rangle. \tag{D.2}$$

If we define

$$|\vec{k}\rangle = \prod_{j=1}^{\infty} \alpha_{-j}^{k_j}|0\rangle, \quad N_{\vec{k}g\vec{Q}} := \sum_{R_1} \chi_{R_1}(C(\vec{k}))\widehat{N}_{R_1 g\vec{Q}}, \tag{D.3}$$

then

$$|R\rangle = \sum_{\vec{k}} \frac{1}{z_{\vec{k}}} \chi_R(C(\vec{k}))|\vec{k}\rangle. \tag{D.4}$$

Thus (D.1) equals

$$\sum_{\vec{k}} \frac{1}{z_{\vec{k}}} \widehat{N}_{\vec{k}g\vec{Q}} \langle \vec{k}| \prod_i \Gamma_-(e^{T_{o,i}}) \prod_i \Gamma_-(e^{T_{e,i}})^{-1}|0\rangle. \tag{D.5}$$

Notice that

$$\begin{aligned} \prod_i \Gamma_-(x_i^{-1})^{\pm} &= \exp \pm \sum_{j=1}^{\infty} \frac{1}{j} \sum_i x_i^j \alpha_{-j} \\ &= \sum_{\vec{k}} \frac{(\pm 1)^{\sum_j k_j}}{z_{\vec{k}}} P_{\vec{k}}(x) \prod_{j=1}^{\infty} \alpha_{-j}^{k_j}. \end{aligned} \tag{D.6}$$

Here $P_{\vec{k}}(x) = \prod_j (\sum_i x_i^j)^{k_j}$. (D.1) becomes

$$\begin{aligned} &\sum_{\vec{k}} \frac{1}{z_{\vec{k}}} \widehat{N}_{\vec{k}g\vec{Q}} \langle \vec{k}| \sum_{\vec{k}_1, \vec{k}_2} \frac{(-1)^{\sum_j k_{2,j}}}{z_{\vec{k}_1} z_{\vec{k}_2}} P_{\vec{k}_1}(e^{-T_o}) P_{\vec{k}_2}(e^{-T_e}) |\vec{k}_1 + \vec{k}_2\rangle \\ &= \sum_{\vec{k}_1, \vec{k}_2} \widehat{N}_{\vec{k}_1 + \vec{k}_2, g, \vec{Q}} \frac{(-1)^{\sum_j k_{2,j}}}{z_{\vec{k}_1} z_{\vec{k}_2}} P_{\vec{k}_1}(e^{-T_o}) P_{\vec{k}_2}(e^{-T_e}). \end{aligned} \tag{D.7}$$

It is clear that the contributions from (D.1) to

$$\sum_{\vec{Q}_b} n_g^{\vec{Q}_b}(X_b) e^{-\vec{Q}_b \cdot \vec{t}} \tag{D.8}$$

have to be symmetric with respect to $e^{-T_{o,i}}$, and also with respect to $e^{-T_{e,i}}$. Any such function can be expanded in $P_{\vec{k}_1}(e^{-T_o}) P_{\vec{k}_2}(e^{-T_e})$, and once we know the coefficients, we can read off $\widehat{N}_{\vec{k}g\vec{Q}}$ using (D.7). Finally one computes the open GV invariants using the formula $\widehat{N}_{Rg\vec{Q}} = \sum_{\vec{k}} \widehat{N}_{\vec{k}g\vec{Q}} \chi_R(C(\vec{k}))/z_{\vec{k}}$.

E. Topological vertex amplitude

We use the convention such that q is replaced by q^{-1} relative to [20]. Explicitly it is given, with slight abuse of notation, by:

$$C_{R_1R_2R_3}(q)=q^{-\frac{1}{2}(\kappa_{R_2}+\kappa_{R_3})}s_{R_2^T}(q^{i-1/2})\sum_Q s_{R_1/Q}(q^{-(R_2^T)_i+i-1/2})s_{R_3^T/Q}(q^{-(R_2)_i+i-1/2}). \tag{E.1}$$

Here s_{R_1/R_2} is a skew Schur function. The index i runs from 1 to ∞.

The partition function of topological strings on any toric Calabi-Yau manifold, with or without D-branes, can be computed by gluing several topological vertices. The gluing rules are explained in subsection 4.2.

F. An identity for geometric transitions

In this appendix we prove the identity (4.8).

First we compute

$$\begin{aligned}
&\sum_R \mathrm{Tr}_{R_5/R}U_{m\times l}(-1)^{|R|}\mathrm{Tr}_{R_6/R^T}U^{-1}_{m\times l}\\
&\quad=\sum_R\langle R_5|\prod_{i=1}^{m}\Gamma_-(q^{-l-m+i-1/2})|R\rangle(-1)^{|R|}\langle R^T|\prod_{i=1}^{m}\Gamma_+(q^{-l-m+i-1/2})|R_6\rangle\\
&\quad=\sum_R\langle R_5|\prod_{i=1}^{m}\Gamma_-(q^{-l-m+i-1/2})|R\rangle\langle R|C\prod_{i=1}^{m}\Gamma_+(q^{-l-m+i-1/2})|R_6\rangle\\
&\quad=(-1)^{|R_6|}\langle R_5|\prod_{i=1}^{m}\Gamma_-(q^{-l-m+i-1/2})\prod_{i=1}^{m}\Gamma_+^{-1}(q^{-l-m+i-1/2})|R_6^T\rangle\\
&\quad=(-1)^{|R_6|}\langle R_5|\prod_{i=1}^{\infty}\Gamma_-(q^{-l-i+1/2})\prod_{i=1}^{\infty}\Gamma_-^{-1}(q^{-l-m-i+1/2})\\
&\qquad\times\prod_{i=1}^{\infty}\Gamma_+^{-1}(q^{-l-m+i-1/2})\prod_{i=1}^{\infty}\Gamma_+(q^{-l+i-1/2})|R_6^T\rangle\\
&\quad=(-1)^{|R_6|}\prod_{i,j=1}^{\infty}(1-q^{m+i+j-1})^{-1}\langle R_5|\prod_{i=1}^{\infty}\Gamma_-(q^{-l-i+1/2})\prod_{i=1}^{\infty}\Gamma_+(q^{-l+i-1/2})\\
&\qquad\times\prod_{i=1}^{\infty}\Gamma_-^{-1}(q^{-l-m-i+1/2})\prod_{i=1}^{\infty}\Gamma_+^{-1}(q^{-l-m+i-1/2})|R_6^T\rangle\\
&\quad=(-1)^{|R_6|}e^{\sum_{n=1}^{\infty}\frac{e^{-ng_sm}}{n[n]^2}}\sum_{R,Q,Q'}\langle R_5|\prod_{i=1}^{\infty}\Gamma_-(q^{-l-i+1/2})|Q\rangle\langle Q|\prod_{i=1}^{\infty}\Gamma_+(q^{-l+i-1/2})|R\rangle\\
&\qquad\times(-1)^{|R|}\langle R^T|\prod_{i=1}^{\infty}\Gamma_-(q^{-l-m-i+1/2})|Q'\rangle\langle Q'|\prod_{i=1}^{\infty}\Gamma_+(q^{-l-m+i-1/2})|R_6\rangle(-1)^{|R_6|}\\
&\quad=e^{\sum_{n=1}^{\infty}\frac{e^{-ng_sm}}{n[n]^2}}\sum_{R,Q,Q'}(-1)^{|R|}q^{l(|R_5|-|R|)}s_{R_5/Q}(q^{i-1/2})s_{R/Q}(q^{i-1/2})\\
&\qquad\times q^{(l+m)(|R|-|R_6|)}s_{R^T/Q'}(q^{i-1/2})s_{R_6/Q'}(q^{i-1/2})\\
&\quad=q^{l|R_5|-(l+m)|R_6|-\frac{1}{2}\kappa_{R_5}-\frac{1}{2}\kappa_{R_6}}e^{\sum_{n=1}^{\infty}\frac{e^{-ng_sm}}{n[n]^2}}\sum_R C_{\cdot R_5^T R}(-1)^{|R|}e^{-|R|g_sm}C_{R^T\cdot R_6^T}.
\end{aligned}\tag{F.1}$$

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Combining this with (3.9) when $R_i = 0$ gives (4.8).

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